

# A NOTE ON THE REGULARITY OF HIBI RINGS

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**ABSTRACT.** We compute the regularity of the Hibi ring of any finite distributive lattice in terms of its poset of join irreducible elements.

## INTRODUCTION

Let  $P$  be a finite poset. The set of poset ideals  $L = \mathcal{I}(P)$ , partially ordered by inclusion, is a distributive lattice. According to a classical result of Birkhoff [1] any finite distributive lattice arises in this way. Now given a field  $K$ , there is naturally attached to  $L$  the  $K$ -algebra  $K[L]$  generated over  $K$  by the elements of  $L$  with defining relations  $\alpha\beta - (\alpha \wedge \beta)(\alpha \vee \beta)$  with  $\alpha, \beta \in L$  incomparable. This algebra was introduced by Hibi [6] in 1987 where he showed that  $K[L]$  is a Cohen–Macaulay domain with an ASL structure. He also characterized those distributive lattices for which  $K[L]$  is Gorenstein. Nowadays  $K[L]$  is called the Hibi ring of  $L$ .

By choosing for each  $\alpha \in L$  an indeterminate  $x_\alpha$  one obtains the presentation  $K[L] \cong S/I_L$  where  $S$  is the polynomial ring over  $K$  in the indeterminates  $x_\alpha$  and where  $I_L$  is generated by the quadratic binomials  $x_\alpha x_\beta - x_{\alpha \wedge \beta} x_{\alpha \vee \beta}$  with  $\alpha, \beta \in L$  incomparable. Not so much is known about the graded minimal free  $S$ -resolution of the toric ideal  $I_L$ . Of course we know its projective dimension. Indeed, since  $K[L]$  is Cohen–Macaulay and since  $\dim K[L]$  is known to be equal to  $|P| + 1$ , the Auslander–Buchsbaum formula implies that  $\operatorname{projdim} I_L = |L| - |P| - 2$ . An equally important invariant of a graded module  $M$  over a polynomial ring is its Castelnuovo–Mumford regularity which may be computed in terms of the shifts of the graded minimal free resolution of  $M$  and which is denoted by  $\operatorname{reg} M$ . As a main result of this paper we show that  $\operatorname{reg} I_L = |P| - \operatorname{rank} P$ . As a consequence we obtain the formula as given in [4] for the regularity of  $I_L$  for any planar distributive lattice  $L$ . Our result also provides a simple proof for the classification of the distributive lattices for which  $I_L$  has a linear resolution, see [3, Theorem 3.2] and [4, Corollary 10], and of those lattices for which  $I_L$  is extremal Gorenstein, see [4, Theorem 3.5].

## 1. THE REGULARITY OF $K[L]$

Let  $P$  be a finite poset. A subset  $\alpha \subset P$  is called a *poset ideal* of  $P$  if whenever  $p \in \alpha$  and  $q \leq p$ , then  $q \in \alpha$ . We denote by  $\mathcal{I}(P)$  the set of poset ideals of  $P$ . Note

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that  $\mathcal{I}(P)$  with the partial order given by inclusion and with union and intersection as join and meet operation is a distributive lattice. Birkhoff's fundamental theorem asserts that any finite distributive lattice  $(L, \wedge, \vee)$  arises in this way. To be precise,  $L \cong \mathcal{I}(P)$  where  $P$  is the subposet of  $L$  consisting of all join irreducible elements of  $L$ . Recall that  $\alpha \in L$  is called *join irreducible* if  $\alpha \neq \min L$  and whenever  $\alpha = \beta \vee \gamma$ , then  $\alpha = \beta$  or  $\alpha = \gamma$ .

Due to this theorem, we may from now on assume  $L = \mathcal{I}(P)$  for some poset  $P$ . This point of view allows us to interpret  $K[L]$  as a toric ring. Indeed, let  $S$  be the polynomial ring over  $K$  in the variables  $x_\alpha$  with  $\alpha \in L$ , and let  $T$  be the polynomial ring over  $K$  in the variables  $s$  and  $t_p$  with  $p \in P$ . We consider the  $K$ -algebra homomorphism  $\varphi: S \rightarrow T$  with  $\varphi(x_\alpha) = s \prod_{p \in \alpha} t_p$ . It is shown in [6] that  $I_L = \text{Ker } \varphi$ . Thus we see that

$$K[L] \cong K[\{s \prod_{p \in \alpha} t_p : \alpha \in L\}] \subset T.$$

We henceforth identify  $K[L]$  with  $K[\{s \prod_{p \in \alpha} t_p : \alpha \in L\}]$ . In [6, (3.2)] Hibi describes the monomial  $K$ -basis of  $K[L]$ : let  $\hat{P}$  be the poset obtained from  $P$  by adding the elements  $-\infty$  and  $\infty$  with  $\infty > p$  and  $-\infty < p$  for all  $p \in P$ , and let  $\mathcal{S}(\hat{P})$  be the set of integer valued functions

$$v: \hat{P} \rightarrow \mathbb{N}$$

with  $v(\infty) = 0$  and  $v(p) \leq v(q)$  for all  $p \geq q$ . Then the monomials

$$(1) \quad s^{v(-\infty)} \prod_{p \in P} t_p^{v(p)}, \quad v \in \mathcal{S}(\hat{P})$$

form a  $K$ -basis of  $K[L]$ . Note that  $K[L]$  is standard graded with

$$(2) \quad \deg(s^{v(-\infty)} \prod_{p \in P} t_p^{v(p)}) = v(-\infty).$$

Let  $\omega_L$  be the canonical ideal of  $K[L]$ . By using a result of Stanley [8, pg. 82], Hibi shows in [6, (3.3)] that the monomials

$$(3) \quad s^{v(-\infty)} \prod_{p \in P} t_p^{v(p)}, \quad v \in \mathcal{T}(\hat{P})$$

form a  $K$ -basis of  $\omega_L$ , where  $\mathcal{T}(\hat{P})$  is the set of integer valued functions  $v: \hat{P} \rightarrow \mathbb{N}$  with  $v(\infty) = 0$  and  $v(p) < v(q)$  for all  $p > q$ .

Based on these facts, we are now ready to prove the following

**Theorem 1.1.** *Let  $L$  be a finite distributive lattice and  $P$  the poset of join irreducible elements of  $L$ . Then*

$$\text{reg } I_L = |P| - \text{rank } P.$$

*Proof.* Let  $H_{K[L]}(t)$  be the Hilbert series of  $K[L]$ . Then

$$H_{K[L]}(t) = \frac{Q(t)}{(1-t)^d},$$

where  $Q(t) = \sum_i h_i t^i$  is a polynomial and where  $d = |P| + 1$  is the Krull dimension of  $K[L]$ . Since  $K[L]$  is Cohen-Macaulay, it follows that  $\text{reg } K[L] = \deg Q(t)$ .

The  $a$ -invariant  $a(K[L])$  of  $K[L]$  is defined to be the degree of the Hilbert series of  $K[L]$  (see [2, Def. 4.4.4]) which by definition is equal to  $\deg Q(t) - d$ . Thus we see that

$$(4) \quad \operatorname{reg} I_L = \operatorname{reg} K[L] + 1 = a(K[L]) + |P| + 2.$$

On the other hand, following Goto and Watanabe [5], who introduced the  $a$ -invariant, we have

$$a(K[L]) = -\min\{i: (\omega_L)_i \neq 0\},$$

see [2, Def. 3.6.13]. Thus, since  $\operatorname{rank} \hat{P} = \operatorname{rank} P + 2$ , the desired formula for the regularity of  $K[L]$  follows from (4) once we have shown that  $\min\{i: (\omega_L)_i \neq 0\} = \operatorname{rank} \hat{P}$ .

Let  $v \in \mathcal{T}(\hat{P})$  and let  $-\infty < p_1 < \cdots < p_r < \infty$  be a maximal chain in  $\hat{P}$  with  $r = \operatorname{rank} P + 1$ . Then

$$0 < v(p_r) < v(p_{r-1}) < \cdots < v(p_1) < v(-\infty).$$

It follows that  $v(-\infty) \geq \operatorname{rank} \hat{P}$ , and hence (3) implies that  $\min\{i: (\omega_L)_i \neq 0\} \geq \operatorname{rank} \hat{P}$ . In order to prove equality, we consider the depth function  $\delta: \hat{P} \rightarrow \mathbb{N}$  which for  $p \in \hat{P}$  is defined to be the supremum of the lengths of chains ascending from  $p$ . Obviously,  $\delta \in \mathcal{T}(\hat{P})$  and  $\delta(-\infty) = \operatorname{rank} \hat{P}$ . This concludes the proof of the theorem.  $\square$

Recall that  $L = \mathcal{I}(P)$  is called *simple* if there is no  $p \in P$  with the property that for every  $q \in P$  either  $q \leq p$  or  $q \geq p$ . In the further discussions we may assume without any restrictions that  $L$  is simple, because if we consider the subposet  $P'$  of  $P$  which is obtained by removing a vertex  $p \in P$  which is comparable with any other vertex of  $P$  and let  $L' = \mathcal{I}(P')$ , then  $I_L$  and  $I_{L'}$  have the same regularity. Indeed,  $|P'| = |P| - 1$ , and since any maximal chain of  $P$  passes through  $p$ , it also follows that  $\operatorname{rank} P' = \operatorname{rank} P - 1$ . Thus the assertion follows from Theorem 1.1.

As an immediate consequence of Theorem 1.1, we get the following characterization of simple distributive lattices whose Hibi rings have linear resolutions, previously obtained in [3] and [4].

**Corollary 1.2.** *Let  $L$  be a finite simple distributive lattice and  $P$  the poset of join irreducible elements of  $L$ . Then  $I_L$  has a linear resolution if and only if  $P$  is the sum of a chain and an isolated element.*

*Proof.* The ideal  $I_L$  has a linear resolution if and only if  $\operatorname{reg} I_L = 2$ . By Theorem 1.1 this is the case if and only if  $|P| - \operatorname{rank} P = 2$ . Say,  $\operatorname{rank} P = r$ , and let  $C = p_0 < p_1 < \cdots < p_r$  be a maximal chain in  $P$ . Thus  $|P| - \operatorname{rank} P = 2$ , if and only if there exists a unique  $q \in P$  not belonging to  $C$ . Suppose  $q$  is comparable with some  $p_i$ . Then  $p_i$  is comparable with any other element of  $P$ , contradiction the assumption that  $L$  is simple. Thus if  $L$  is simple, then  $|P| - \operatorname{rank} P = 2$  if and only if  $P$  is the sum of the chain  $C$  and the isolated element  $q$ .  $\square$

The preceding corollary implies that a finite simple distributive lattice is planar if  $I_L$  has a linear resolution. Now let  $L$  be any simple planar lattice and  $P$  the poset

of join irreducible elements of  $L$ . Then there exist two chains  $C_1$  and  $C_2$  such that  $P$  as a set is the disjoint union of them. We may assume that  $|C_1| \geq |C_2|$ . It follows from Theorem 1.1 that  $\text{reg } I_L = |C_2| + 1$ . This result may also be obtained with the characterization given in [4, Theorem 4].

We would like to remark that, given a number  $k$ , Theorem 1.1 allows us to determine in a finite number of steps all finite simple distributive lattices  $L$  with  $\text{reg } I_L = k$ . As an example, we consider the case  $k = 3$ . Let  $P$  be the poset of join irreducible poset of  $L$ . By Theorem 1.1, it is enough to find all finite posets  $P$  with  $|P| - \text{rank } P = 3$ . Let  $C$  be a maximal chain in  $P$ . Since  $|P| = \text{rank } P + 3$ , it follows that there exist precisely two elements  $q, q' \in P$  which do not belong to  $C$ . The only posets satisfying  $|P| = \text{rank } P + 3$  for which  $L = \mathcal{I}(P)$  is simple are displayed in Figure 1.

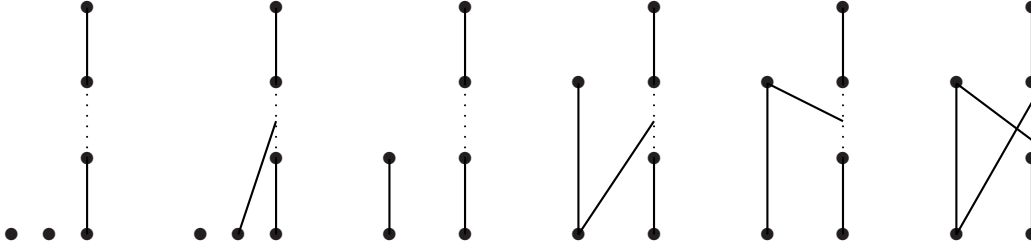


FIGURE 1.

The Gorenstein ideals  $I_L$  with  $\text{reg } I_L = 3$  are called *extremal Gorenstein*. Hibi showed in [6, pg. 105, d) Corollary] that for any distributive lattice  $L$ , the ideal  $I_L$  is Gorenstein if and only if the poset of join irreducible elements of  $L$  is pure. Combining this fact with the above consideration, we recover the result of [3, Theorem 3.5] which says that for a simple distributive lattice  $L$ , the ideal  $I_L$  is extremal Gorenstein if and only if  $L$  is one of the lattices shown in Figure 2.

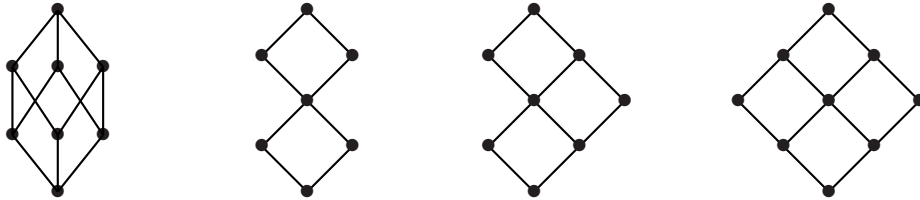


FIGURE 2.

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